

Ordinal ultrafilters versus P-hierarchy

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Abstract

An earlier paper, entitled “P-hierarchy on $\beta\omega$ ”, investigated the relations between ordinal ultrafilters and the so-called P-hierarchy. The present paper focuses on the aspects of characterization of classes of ultrafilters of finite index, existence, generic existence and the Rudin-Keisler-order.

1 Introduction

Ultrafilters on ω may be classified with respect to sequential contours of different ranks, that is, iterations of the Fréchet filter by contour operations. This way an ω_1 -sequence $\{\mathcal{P}_\alpha\}_{1 \leq \alpha \leq \omega_1}$ of pairwise disjoint classes of ultrafilters - the P-hierarchy - is obtained, where P-points correspond to the class \mathcal{P}_2 , allowing us to look at the P-hierarchy as an extension of P-points. Section 2 recalls all necessary definitions and properties of the P-hierarchy. Section 3 shows some equivalent conditions for an ultrafilter to belong to a class of (fixed) finite index of the P-hierarchy; those conditions appear to be very similar to the behavior of classical P-points. We also obtain another condition for belonging to a class of (fixed) finite index of the P-hierarchy which is literally a part of conditions for being an element of a class of (fixed) finite index of ordinal ultrafilters. Section 4 focuses on the Rudin-Keisler order on P-hierarchy classes. It is shown that RK minimal elements of classes of finite index can exist. Similar results are achieved for ordinal ultrafilters. In section 5 we show evidence for the generic existence of the P-hierarchy being equivalent to $\mathfrak{d} = \mathfrak{c}$, in consequence, being equivalent to the generic existence

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of ordinal ultrafilters. In section 6 we prove that CH implies that each class of the P-hierarchy is not empty, we also presented known results concerning existence of both types of ultrafilters.

We generally use standard terminology, however less popular terms are taken from [6], where key-term "monotone sequential cascades" has been introduced. All necessary information may also be found in [18]. For additional information regarding sequential cascades and contours a look at [6], [7], [5], [16] is recommended. Below, only the most important definitions and conventions are repeated.

If u is a filter on $A \subset B$, then we identify u with the filter on B for which u is a filter-base.

Let p be a filter on X , and let q be a filter on Y ; we say that p is *Rudin-Keisler* greater than q (we write $p \geq_{RK} q$) if there is such a map $f : X \rightarrow Y$ that $f(p) \supset q$. We say that p is *infinite Rudin-Keisler* greater than q (we write $p >_\infty q$) if there is a map $f : X \rightarrow Y$ with $f(p) = q$, but there is no $P \in p$ such that $f|_P$ is finite-to-one. We say that p is *greater* than q if $q \subset p$.

Recall also that if p, q are ultrafilters, $f(p) = q$ and if $p \approx q$ (i.e. $p \geq_{RK} q$ and $q \geq_{RK} p$), then there exists $P \in p$ such that $f|_P$ is one-to-one (see [3, Theorem 9.2]).

The *cascade* is a well founded tree i.e. a tree V without infinite branches and with a least element \emptyset_V . A cascade is *sequential* if for each non-maximal element of V ($v \in V \setminus \max V$) the set v^{+V} of immediate successors of v (in V) is countably infinite. For $v \in V$ we write v^{-V} to denote such an element of V that $v \in (v^{-V})^{+V}$. For $A \subset V$ we use $A^{+V} = \bigcup_{v \in A} v^{+V}$, $A^{-V} = \bigcup_{v \in A} v^{-V}$. In symbols v^{+V} , v^{-V} , A^{+V} , A^{-V} we omit the name of cascade (obtaining v^+ , v^- , A^+ , A^-) if it is clear from the context which cascade we have on mind. If $v \in V \setminus \max V$, then the set v^+ (if infinite) may be endowed with an order of the type ω , and then by $(v_n)_{n \in \omega}$ we denote the sequence of elements of v^+ , and by v_{nV} - the n -th element of v^{+V} .

The *rank* of $v \in V$ ($r_V(v)$ or $r(v)$) is defined inductively as follows: $r(v) = 0$ if $v \in \max V$, and otherwise $r(v)$ is the least ordinal greater than the ranks of all immediate successors of v . The rank $r(V)$ of the cascade V is, by definition, the rank of \emptyset_V . If it is possible to order all sets v^+ (for $v \in V \setminus \max V$) so that for each $v \in V \setminus \max V$ the sequence $(r(v_n)_{n < \omega})$ is non-decreasing, then the cascade V is *monotone*, and we fix such an order on V without indication.

Let W be a cascade, and let $(V_w)_{w \in \max W}$ be a pairwise disjoint sequence of cascades such that $V_w \cap W = \emptyset$ for all $w \in \max W$. Then, the *confluence* of cascades V_w with respect to the cascade W (we write $W \leftarrow P V_w$) is defined as a cascade constructed by the identification $w \in \max W$ with \emptyset_{V_w} and according to the following rules: $\emptyset_W = \emptyset_{W \leftarrow P V_w}$; if $w \in W \setminus \max W$, then

$w^{+W \leftrightarrow V_w} = w^{+W}$; if $w \in V_{w_0}$ (for a certain $w_0 \in \max W$), then $w^{+W \leftrightarrow V_w} = w^{+V_{w_0}}$; in each case we also assume that the order on the set of successors remains unchanged. By $(n) \leftrightarrow V_n$ we denote $W \leftrightarrow V_w$ if W is a sequential cascade of rank 1.

If $\mathbb{P} = \{p_s : s \in S\}$ is a family of filters on X and if q is a filter on S , then the *contour of $\{p_s\}$ along q* is a filter on X defined by

$$\int_q \mathbb{P} = \int_q p_s = \bigcup_{Q \in q} \bigcap_{s \in Q} p_s.$$

Such a construction has been used by many authors ([8], [9], [10]) and is also known as a sum (or as a limit) of filters. On the sequential cascade, we consider the finest topology such that for all but the maximal elements v of V , the co-finite filter on the set v^{+V} converges to v . For the sequential cascade V we define the *contour* of V (we write $\int V$) as the trace on $\max V$ of the neighborhood filter of \emptyset_V (the *trace* of a filter u on a set A is the family of intersections of elements of u with A). Equivalently we may say that $\int V$ is a Fréchet filter on $\max V$ if $r(V) = 1$, and $\int V = \int_{Fr} \int V_n$ if $V = (n) \leftrightarrow V_n$ where Fr denotes the Fréchet filter. Similar filters were considered in [11], [12], [4]. Let V be a monotone sequential cascade and let $u = \int V$. Then the *rank* $r(u)$ of u is, by definition, the rank of V . It was shown in [7] that if $\int V = \int W$, then $r(V) = r(W)$.

Let S be a countable set. A family $\{u_s\}_{s \in S}$ of filters is referred to as *discrete* if there exists a pairwise disjoint family $\{U_s\}_{s \in S}$ of sets such that $U_s \in u_s$ for each $s \in S$. For $v \in V$ we denote by v^\uparrow a subcascade of V built by v and all successors of v . If $U \subset \max V$ and $U \in \int V$, then by $U^\downarrow V$ we denote the biggest (in the set-theoretical order)¹⁾ monotone sequential subcascade of the cascade V built of some $v \in V$ such that $U \cap \max v^\uparrow \neq \emptyset$. We write v^\uparrow and U^\downarrow instead of $v^{\uparrow V}$ and $U^{\downarrow V}$ if we know in which cascade the subcascade is considered. By V_n we usually denote $(\emptyset_V)_n^\uparrow$, by $V_{n,m}$ we understand $((\emptyset_V)_n^+)_m^\uparrow$.

2 Ordinal ultrafilters and classes \mathcal{P}_α

In the remainder of this paper each filter is considered to be on ω , unless otherwise indicated. Let us define \mathcal{P}_α for $1 \leq \alpha < \omega_1$ on $\beta\omega$ (see [18]) as follows: $u \in \mathcal{P}_\alpha$ if there is no monotone sequential contour \mathcal{V}_α of rank α such that $\mathcal{V}_\alpha \subset u$, and for each β in the range $1 \leq \beta < \alpha$ there exists a monotone

i.e. the cascade V is greater than the cascade W if $V \subset W$, $\emptyset_V = \emptyset_W$, $w^{+W} \subset w^{+V}$ for all $w \in W$, $\max W \subset \max V$.

sequential contour \mathcal{V}_β of rank β such that $\mathcal{V}_\beta \subset u$. Moreover, if for each $\alpha < \omega_1$ there exists a monotone sequential contour \mathcal{V}_α of rank α such that $\mathcal{V}_\alpha \subset u$, then we write $u \in \mathcal{P}_{\omega_1}$.

Let us recall three equivalent definitions of P-points: a point $u \in \beta\omega \setminus \omega$ is a *P-point* if

- A) the intersection of countably many neighborhoods of u is a (not necessarily open) neighborhood of u ;
- B) for each countable set $\{U_n\}_{n < \omega}$ of elements of the ultrafilter u there exists a set $U \in u$ such that $\text{card}(U \setminus U_n)$ is finite for each $n < \omega$;
- C) for each function $f : \omega \rightarrow \omega$ there exists a set $U \in u$ such that either $f|_U$ is constant or $f|_U$ is finite-to-one.

Remark 2.1. *If u is an ultrafilter on ω then:*

- 1) $u \in \mathcal{P}_1$ if and only if u is a principal ultrafilter;
- 2) if u is RK-minimal then $u \in \mathcal{P}_2$. ■

Let M be a countably infinite set, and let V be a monotone cascade of rank $\alpha < \omega_1$ such that $\max V = M$. Then the set $D = \{D_v = \max v^\uparrow : v \in V, r(v) \geq 1\}$ is called an α -partition (of M).

Thus, the classic “partitions of ω into infinitely many infinite sets” belong to “2-partitions” in our language. Since a cascade uniquely defines a partition, it is usually identified with its cascade. For an α -partition we define by transfinite induction *residual sets* as follows: a set A is residual for a 1-partition V if $A \cap \max V$ is finite; if residual sets are defined for all β -partitions for $\beta < \alpha$, then a set A is residual for the α -partition $V = (n) \dot{\leftrightarrow} V_n$ if there exists a finite set $N \subset \omega$ such that for all $n \notin N$ the set A is residual for the partitions V_n . For a partition defined by a monotone sequential cascade V , equivalently we can say that U is residual if and only if $\omega \setminus U \in \int V$.

Certain properties of the P-hierarchy from [18] are listed below, namely Proposition 2.1 and Theorems 2.3, 2.5, 2.9, 2.8.

Proposition 2.2. *An ultrafilter u (on ω) is a P-point if and only if $u \in \mathcal{P}_2$.*

Theorem 2.3. *Let $u \in \mathcal{P}_\alpha$ and let $f : \omega \rightarrow \omega$. Then $f(u) \in \mathcal{P}_\beta$ for a certain $\beta \leq \alpha$.*

Let α be an ordinal, by $-1 + \alpha$ we understand $\alpha - 1$ if α is finite, and α if α is infinite.

Theorem 2.4. *Let $(\alpha_n)_{n < \omega}$ be a non-decreasing sequence of ordinals less than ω_1 , let $\alpha = \lim_{n < \omega} (\alpha_n)$, let $1 < \beta < \omega_1$ and let (X_n) be a partition of ω . If (p_n) is a sequence of ultrafilters such that $X_n \in p_n \in \mathcal{P}_{\alpha_n}$ and $p \in \mathcal{P}_\beta$, then $\int_p p_n \in \mathcal{P}_{\alpha + (-1 + \beta)}$.*

Theorem 2.5. *Let α, β be countable ordinals. If $u \in \mathcal{P}_{\alpha+\beta+1}$ then there exists a function $f : \omega \rightarrow \omega$ such that $f(u) \in \mathcal{P}_{\beta+1}$.*

Theorem 2.6. *The following statements are equivalent:*

- 1) *P -points exist,*
- 2) *the \mathcal{P}_α classes are non-empty for each countable successor α ,*
- 3) *There exists a countable successor $\alpha > 1$ such that the class \mathcal{P}_α is non-empty.*

In [1] Baumgartner provides the following definition. Let I be a family of subsets of a set A such that I contains all singletons and is closed under subsets. Given an ultrafilter u on ω , we say that u is an I -ultrafilter if for any $f : \omega \rightarrow A$ there is $U \in u$ such that $f(U) \in I$. For $\alpha < \omega_1$, let $I_\alpha = \{B \subset \omega_1 : B \text{ has an order type } \leq \alpha\}$, $J_\alpha = \{B \subset \omega_1 : B \text{ has order type } < \alpha\}$. A *proper* I_α -ultrafilter is one which is not an I_β -ultrafilter for any $\beta < \alpha$. Denote also proper J_α -ultrafilters as J_α^* -ultrafilters those being the J_α -ultrafilters which are not J_β -ultrafilters for any $\beta < \alpha$. We can also find in [1] the following statement: If u is a proper I_α -ultrafilter, then α must be indecomposable. Recall that

Proposition 2.7. *[18, a corollary of Proposition 3.3] If $u \in J_{\omega_\alpha}^*$, then $u \in \mathcal{P}_\beta$ for a certain $\beta \leq \alpha$.*

3 \mathcal{P}_α classes for finite α and $<_\infty$ sequences

Theorem 3.1. *If $u \in \beta\omega$, then the following statements are equivalent:*

- 1) *There is no monotone sequential contour C of rank n such that $C \subset u$. (i.e., for each n -partition there exists a set $U \in u$ residual for this partition)*
- 2) $u \in \bigcup_{i=1}^n \mathcal{P}_i$.
- 3) *For each family of functions $\{f_1, \dots, f_{n-1}\}$, $f_i : \omega \rightarrow \omega$ there exists a set $U \in u$ such that*
 - a) $f_1 \circ \dots \circ f_{n-1} \upharpoonright_U$ *is constant or*
 - b) *there exists $i \in \{1, \dots, n-1\}$ such that $f_i \upharpoonright_{f_{i+1} \circ \dots \circ f_{n-1}(U)}$ is finite-to-one.*
- 4) *For each function $f : \omega \rightarrow \omega$ there exists a set $U \in u$ such that*
 - a) $f^{n-1} \upharpoonright_U$ *is constant or*
 - b) *there exists $i \in \{1, \dots, n-1\}$ such that $f \upharpoonright_{f^{i-1}(U)}$ is finite-to-one.*

Proof. $1 \Leftrightarrow 2$ is trivial.

$2 \Rightarrow 3$: Let $u \in \mathcal{P}_i$ for some $i \leq n$ and let us take any functions $f_1, \dots, f_{n-1} : \omega \rightarrow \omega$.

Let $A_k^\infty = \{m < \omega : \text{card}(f_k^{-1}(m)) = \omega\}$ and $A_k^{fin} = \{m < \omega : \text{card}(f_k^{-1}(m)) < \omega\}$ for $k \in \{1, \dots, n-1\}$. Since u is an ultrafilter, and $A_k^\infty \cup A_k^{fin} = \omega$, for each k one of those sets belongs to $f_{k+1} \circ \dots \circ f_{n-1}(u)$. If for some k it is A_k^{fin} , then case 3b) holds, so we can assume that for each k , each function f_k is infinite-to-one on elements of $f_{k+1} \circ \dots \circ f_{n-1}(u)$. Since our research is restricted to elements of images of u , without loss of generality we may assume that $\text{card}(f_k^{-1}(m)) = \omega$ for each $k \in \{1, \dots, n-1\}$ and for each $m \in \omega$.

Note the following obvious claim: Let u be an ultrafilter and let f be a function such that $f^{-1}(n)$ is infinite for all $n < \omega$. Then for each monotone sequential cascade V of rank α such that $\int V \subset f(u)$, there is $\int f^{-1}(V) \subset u$, and $r(f^{-1}(v)) = 1 + \alpha$, where $f^{-1}(V) = V \dot{\leftarrow} f^{-1}(v)$.

If $f_1 \circ \dots \circ f_{n-1}(u)$ is not a principal ultrafilter, then $f_k \circ \dots \circ f_{n-1}(u)$ contain a contour of rank k , and thus u contain a contour of rank n - a contradiction.

$3 \Rightarrow 4$ is trivial.

$4 \Rightarrow 1$: Let us assume that there exists a monotone sequential contour C_n of rank n such that $C_n \subset u$. There exists a monotone sequential cascade V such that $\int V = C_n$. Naturally, $r(V) = n$. Without loss of generality we may assume that $\max V = \omega$ and the cascade V is complete, i.e. each branch has the same length n . We identify elements of $\max V$ with n -sequences of natural numbers which label that elements i.e. $\emptyset_V = \emptyset$, $(i_1, \dots, i_k)_{i_{k+1}}^+ = (i_1, \dots, i_{k+1})$. We define the function $f : \max V \rightarrow \max V$ as follows:

$f((i, 1, 1, \dots, 1, 1)) = (1, 1, \dots, 1, 1)$ for each $i < \omega$;

if $v = (k_1, \dots, k_n)$ and if there exists $l \in \{2, \dots, n\}$ such that $k_l \neq 1$ then let $m(v) = \min \{t \in \{1, \dots, n\} : \forall l > t, l \leq n: k_l = 1\}$ and let $f(v) = (k_1, \dots, k_{m(v)-2}, k_{m(v)-1} + 1, 1, \dots, 1, 1)$.

It may be noticed without difficulty that $f^i(\max V) = \{v \in \max V : m(v) < n - i\}$ for $i \in \{0, \dots, n-1\}$. Let $V(i) = (f^i(\max V))^{\downarrow V}$.

Take any $G \subset f^i(\max V)$. If f is finite-to-one on G , then $G \cap \max v^{\uparrow V(i)}$ is finite for each $v \in V(i)$ such that v is a sequence of length $n - i$. Thus, G is residual for $V(i)$ and so does not belong to $f^i(u)$.

Thus, the ultrafilter u does not have the property described in point 4 of Theorem 3.1. ■

It is worth comparing the definitions of P-points from page 3 with the conditions of Theorem 3.1 in order to see that the behavior of P-points is, in a very natural way, extended onto the behavior of elements of $\bigcup_{i=1}^n \mathcal{P}_i$. Condition 1 of Theorem 3.1 is the extension of the equivalent definition of P-point from Theorem 2.3, Condition 2 can be expressed as “ u is no more than \mathcal{P}_n -point”,

and Conditions 3 and 4 of Theorem 3.1 extend the definition “C” of P-point from page 3.

Proposition 3.2. *Let $u \in \mathcal{P}_n$, $n \in \omega$, and $f : \omega \rightarrow \omega$. If $f(u) \in \mathcal{P}_n$ then there exists a set $U \in u$ such that $f \upharpoonright_U$ is finite-to-one.*

Proof. Proof basis on the same idea as a proof of $2 \Rightarrow 3$ in the previous Theorem 3.1. Suppose on the contrary, that there is no $U \in u$ that $f \upharpoonright_U$ is finite-to-one, thus $\{i < \omega : \text{card}(f^{-1}(i)) = \omega\} \in f(u)$, and so without loss of generality we may assume that $\{i < \omega : \text{card}(f^{-1}(i)) = \omega\} = \omega$. If $f(u) \in \mathcal{P}_n$, then there exists a monotone sequential contour $\mathcal{V} \subset f(u)$ of rank $n - 1$. Consider a monotone sequential cascade V such that $\int V = \mathcal{V}$ and $W = V \dot{\cup} f^{-1}(v)$, where $v \in \max V$. Since W is a cascade of rank n and $u \in \mathcal{P}_n$, there exists a set $U \in u$ residual for W . Consider sets W^i for $i \in \{0, \dots, n-1\}$ defined by $W_0 = U$ and $w \in W^{i+1} \Leftrightarrow \text{card}(w^{+W} \cap W^i) = \omega$ (sets W^i are subsets of W and of V as well, for $i > 0$). Split U into n pieces: $U_{n-1} = (W^{n-1})^{\uparrow W} \cap U$, $U^{i-1} = ((W^{i-1})^{\uparrow W} \cap U) \setminus \bigcup_{j=i}^n U_j$. Notice that $U_i \notin u$ for $i > 0$. Indeed, $f(U_i) \subset (W^i)^{\uparrow V} \cap \max V$ and $(W^i)^{\uparrow V} \cap \max V$ is residual for V . Thus $U_0 \in u$, clearly $f \upharpoonright_{U_0}$ is finite-to-one. ■

By Proposition 3.2 and Theorems 2.3 and 2.5 we obtain the following

Corollary 3.3. *If u is an ultrafilter, then $u \in \mathcal{P}_n$ if and only if there exists an n -element $<_\infty$ -decreasing sequence below it that contain “ u ” and a principal ultrafilter, and there is no such chain of length $n + 1$.*

Proof. Non existence of such $n + 1$ chain follows from Proposition 3.2. Existence of n chain follows inductively from a following fact: If $\int V \subset u$ for monotone sequential cascade V , then $\int f(V) \subset f(u)$ where $f[v^+] = v_1$ for all $v \in V : r(v) = 1$, and f is an identity on the rest of V - such defined f is not finite-to-one on each $U \in u$ (for details see proof of Proposition 3.5). Note that if $r(V)$ is finite then $r(f(V)) = r(V) - 1$. ■

In [14] Laflamme shows:

Proposition 3.4. *[14, Reformulation of Proposition 2.3]*

Let $k \in \omega$ and u an ultrafilter such that

() $(\forall h \in {}^\omega \omega_1)(\exists X \in u)$ the order type of $h(X)$ is strictly less than ω^ω*

Then u is an $J_{\omega, k}^$ -ultrafilter precisely if it has a $<_\infty$ -chain of length k below it that contain u and a principal ultrafilter but no such a chain of length $k + 1$.*

Notice that Proposition 3.4 for ordinal ultrafilters is very similar to Corollary 3.3, the only difference being the extra assumption (*).

As opposed to Proposition 3.2, for infinite α 's we have the following

Proposition 3.5. *If α is a countably infinite successor ordinal, then for each $u \in \mathcal{P}_\alpha$ there exists a function $f : \omega \rightarrow \omega$ such that: $f(u) \in \mathcal{P}_\alpha$ and $f|_U$ is not finite-to-one for each $U \in u$.*

Proof. Let $u \in \mathcal{P}_\alpha$, let α be as in the assumptions. Let us take a monotone sequential contour \mathcal{V} of rank $\alpha - 1$ such that $\mathcal{V} \subset u$; consider a monotone sequential cascade V such that $\int V = \mathcal{V}$; without loss of generality we may assume that $\max V = \omega$. For each $v \in V$ such that $r(v) = 1$ choose $\tilde{v} \in v^\uparrow$ and define $f : \max V \rightarrow \omega$ as follows: if $v \in w^\uparrow$ for $w \in V$, $r(w) = 1$ then $f(v) = \tilde{w}$.

We will prove that the function f fulfils the claim. Clearly, the function f is not constant on any $U \in u$. Consider $T = \{v \in V : r(v) = \omega\}$. It is sufficient to prove that $r(f(v^\uparrow)) = \omega$ for each $v \in T$. Let $v^\uparrow = (n) \leftarrow (v_n)^\uparrow$ for $v \in T$. We have $r(\int (v_n)^\uparrow) = r(f(\int (v_n)^\uparrow)) + 1$, so $\lim_{n < \omega} r(f(\int (v_n)^\uparrow)) = \omega = \lim_{n < \omega} r(\int (v_n)^\uparrow)$, and so $r(f(\int V)) = r(\int V) = \alpha - 1$.

Suppose that $f|_U$ is finite-to-one for some $U \in u$. Then $\omega \setminus U \in \int V$, contradiction with $\int V \subset u$.

On the other hand by Theorem 2.3, $f(u) \in \mathcal{P}_\gamma$ for a certain $\gamma \leq \alpha$. ■

Theorem 3.6. *If α is a countably infinite successor ordinal and $u \in \mathcal{P}_\alpha$, then there exists a function $f : \omega \rightarrow \omega$ such that:*

- 1) $f^n|_U$ is not finite-to-one for any $n \in \omega$ and any $U \in f^{n-1}(u)$ ($f^0(u) = u$),
- 2) the sequence $(f^n)_{n < \omega}$ is (pointwise) convergent;
- 3) $f^n(u) \in \mathcal{P}_\alpha$ for each $n < \omega$, and $(\lim_{n < \omega} f^n)(u) \in \mathcal{P}_\alpha$.

Proof. Let V be a monotone sequential cascade of rank $\alpha - 1$ such that $\int V \subset u$.

Let $T = \{t \in V : r(t) = \omega\}$. Without loss of generality we may assume that for each $v \in T$, for all $n < \omega$ each branch of v_n^\uparrow has length $r(v_n)$. For each $v \in V$ take a non decreasing sequence $(a_n^v)_{n < \omega}$ of natural numbers, such that $a_n^v \leq r(v_n)$, $\lim_{n \rightarrow \infty} a_n^v = \omega$, $\lim_{n \rightarrow \infty} (r(v_n) - a_n^v) = \omega$.

For each pair (v, n) where $V \in T$, $n < \omega$ take a set $T_{v,n} = \{t \in v_n^\uparrow : r(t) = a_n^v\}$. For each $t \in T_{v,n}$ take a function $f_{v,n} : \max t^\uparrow \rightarrow \max t^\uparrow$ defined like in the proof of case 4 \Rightarrow 1 in Theorem 3.1, and glue all this functions in a function $f : \max V \rightarrow \max V$ which satisfies a claim. ■

4 Relatively RK- α -minimal points.

Recall that a free ultrafilter $u \in \beta\omega$ is RK-minimal ¹⁾ if for each $f : \omega \rightarrow \omega$ either $f(u)$ is a principal filter or $f(u) \approx u$. The existence of RK-minimal

¹⁾ also known as Ramsey ultrafilters or selective ultrafilters.

points is independent of ZFC (see [3], [15]). The following theorem describes some properties of RK-minimal points.

Theorem 4.1. (see [3], [15]) *The following statements are equivalent for a free ultrafilter u on ω .*

- 1) u is RK-minimal;
- 2) For each function $f : \omega \rightarrow \omega$ there exists a set $U \in u$ such that either $f|_U$ is constant or $f|_U$ is one-to-one;
- 3) u is a P-point and for each finite-to-one function $f : \omega \rightarrow \omega$ there exists a set $U \in u$ such that $f|_U$ is one-to-one;
- 4) For each partition $d = \{d_n; n < \omega\}$ either there exists a set $U \in u$ such that $\text{card}(U \cap d_n) \leq 1$ for each $n < \omega$, or there exists n_0 such that $d_{n_0} \in u$.

An ultrafilter $u \in \mathcal{P}_\alpha$ is referred to as *relatively RK- α -minimal* if for each $f : \omega \rightarrow \omega$ there is either $u \approx f(u)$ or $f(u) \in \mathcal{P}_\beta$ for a certain $\beta < \alpha$; *relatively $<_\infty$ - α -minimal* ultrafilters are those $u \in \mathcal{P}_\alpha$ which each not finite-to-one (on each set $U \in u$) image is not in \mathcal{P}_α .

The following two propositions, admitting straightforward proofs, are useful in investigations of images of contours.

Proposition 4.2. *If (p_n) is a sequence of filters, p is a filter and $f : \omega \rightarrow \omega$ is a function, then $f(\int_p p_n) = \int_{F(p)} o_n$, where (o_n) is a sequence (possibly a finite sequence) of filters such that $o_i \neq o_j$ for $i \neq j$ and $\{o_j : j < \omega\} = \{f(p_n) : n < \omega\}$, $F(n) = i$ iff $f(p_n) = o_i$.*

Notice that F depends on the order on the set $f(\{p_n : n < \omega\})$, so in the remainder of this paper a function F for f is an arbitrary (but fixed) function among such functions.

Proposition 4.3. *Let (p_n) , p , (o_n) and F be as in Proposition 4.2. Suppose that there exists a set $P \in F(p)$ such that the sequence $(o_i)_{i \in P}$ is discrete and there exists a set $H \in p$ such that $F|_H$ is one-to-one and $p_n \approx o_{F(n)}$ for each $n \in H$. Then $\int_p p_n \approx \int_{F(p)} o_i = f(\int_p p_n)$.*

Theorem 4.4. *Let $m < \omega$. If (p_n) is a discrete sequence of relatively RK- m -minimal free ultrafilters on ω and p is an RK-minimal free ultrafilter, then $\int_p p_n$ is relatively RK- $m + 1$ -minimal.*

Proof. Let p and (p_n) be as in the assumptions. Let f be a function $f : \omega \rightarrow \omega$. By Theorem 2.4 $\int_p p_n \in \mathcal{P}_{m+1}$. Take (o_i) and F as in Proposition 4.2. Thus, $f(\int_p p_n) = \int_{F(p)} o_i$. Without loss of generality we may assume that $\int_{F(p)} o_i \in \mathcal{P}_{m+1}$. We want to prove that $\int_{F(p)} o_i \notin \mathcal{P}_{m+1}$ or $\int_{F(p)} o_i \approx \int_p p_n$. For this and consider two cases:

Case 1. $F(p)$ is a principal filter. In this case there exists $i < \omega$ such that $\{i\} \in F(p)$ and thus $o_i = \int_{F(p)} o_i$. Since $o_i = f(p_j)$ for some $j < \omega$, and by Theorem 2.3 $o_i \in \mathcal{P}_\beta$ for some $\beta \leq m$, we have $f(\int_p p_n) \notin \mathcal{P}_{m+1}$.

Case 2. $F(p)$ is a free filter. Then $F(p)$ is a free ultrafilter, and thus, $F(p) \approx p$, since p is RK-minimal. Define sets $D_i = \{n < \omega : o_n \in \mathcal{P}_i\}$, for $i \in \{1, \dots, m\}$. Since $o_n \in \bigcup_{j=1}^m \mathcal{P}_j$, there exists exactly one $i_0 \in \{1, \dots, m\}$ such that $D_{i_0} \in F(p)$.

Subcase 2.1. $i_0 < m$. Let us take a discrete sequence (q_i) of ultrafilters such that $q_i \approx o_i$, in this aim consider a partition $(A_i)_{i < \omega}$ of ω into infinite sets, and a sequence (f_n) of bijections $f_i : \omega \rightarrow A_i$ and put $q_i = f_i(o_i)$. By Theorem 2.4 $\int_{F(p)} q_i \in \mathcal{P}_{i_0+1}$ there is $\int_{F(p)} o_i \preceq \int_{F(p)} q_n$ and so $\int_{F(p)} o_i \notin \mathcal{P}_{m+1}$.

Subcase 2.2. $i_0 = m$. For each $i \in D_m$ and for each n with $F(n) = i$ we have $p_n \approx f(p_n) = o_{F(n)} = o_i$ by RK-minimality of p_n .

Let $i_1 = \min D_m$. There exists a set A_1 such that $A_1 \in o_{i_1}$ and $A_1 \notin \int_{F(p)} o_i$ (because we are not in case 1). If numbers i_r and sets A_r for $r < t$ are already defined, we define $i_t = \min \{i \in D_m : (\bigcup_{r=1}^{t-1} A_r)^c \in o_i\}$, and let A_t be such a set that $A_t \subset (\bigcup_{r=1}^{t-1} A_r)^c$, $A_t \in o_{i_t}$, $(A_t)^c \in \int_{F(p)} o_i$ (such a set exists because we are not in case 1, and $(\bigcup_{r=1}^{t-1} A_r)^c \in o_{i_t}$). In this way we obtain a sequence $(A_r)_{r < \omega}$ of pairwise disjoint sets such that $(A_r)^c \in \int_{F(p)} o_i$ for each $r < \omega$, and for each $i < \omega$ there exists a number $r < \omega$ such that $A_r \in o_i$. Thus, the sequence $(A_r)_{r < \omega}$ defines a partition $s = (S_n)_{n < \omega}$ of D_m into non-empty sets by letting $i \in S_n$ if and only if $A_n \in o_i$. There is no n such that $S_n \in F(p)$ and $F(p)$ is RK-minimal, so by Theorem 4.1 there exists a set $P \in F(p)$ with $P \subset D_m$ such that $\text{card}(P \cap S_n) \leq 1$ for each $n \in \omega$ (the sequence $(o_i)_{i \in P}$ is discrete). The same Theorem 4.1 shows that there exists a set $H \in p$ such that $F|_H$ is one-to-one. Without loss of generality $F(H) \subset D_m$ and since $P_i \approx o_{F(i)}$ for all $i \in H$ we are in the assumption of Proposition 4.3 so we conclude $\int_{F(p)} o_i \approx \int_p p_n$. ■

By induction, by Theorem 4.4 one can easily prove the following Corollary 4.5:

Corollary 4.5. *If there exist RK-minimal ultrafilters in $\beta\omega \setminus \omega$, then for each $n < \omega$ there exist relatively RK- n -minimal ultrafilters.*

In contrast to the above Corollary 4.5, for infinite α 's we obtain the following from Theorem 3.6:

Corollary 4.6. *There are no $<_\infty$ (and so no RK) relatively minimal ultrafilters in classes of infinite successor index of the P -hierarchy.*

Problem 1. *Do relatively RK- α -minimal ultrafilters exist for limit ordinals $\alpha \leq \omega_1$?*

We may also consider RK-minimal elements in classes of ordinal ultrafilters. An ultrafilter $u \in J_{\omega^\alpha}^*$ is referred to as a *relatively ordinal RK- α -minimal* if for each $f : \omega \rightarrow \omega$ either $u \approx f(u)$ or $f(u) \in J_{\omega^\beta}^*$ for a certain $\beta < \alpha$.

One can get a very similar result to Theorem 4.4 for ordinal ultrafilters:

Theorem 4.7. *Let $m < \omega$. If (p_n) is a discrete sequence of relatively ordinal RK- m -minimal free ultrafilters on ω and p is a RK-minimal free ultrafilter, then $\int_p p_n$ is relatively ordinal RK- $m + 1$ -minimal.*

The proof is very similar to the proof of Theorem 4.4 and uses the following reformulation of a theorem of Baumgartner [1, Theorem 4.2]:

Theorem 4.8. *Let $(\alpha_n)_{n < \omega}$ be a non-decreasing sequence of ordinals less than ω_1 , let $\alpha = \lim_{n < \omega} (\alpha_n)$ and let (X_n) be a partition of ω . If (p_n) is a sequence of ultrafilters such that $X_n \in p_n \in J_{\omega^{\alpha_n}}^*$ and p is a P -point, then $\int_p p_n \in J_{\omega^{\alpha+1}}^*$.*

Notice also that, by Proposition 3.2, each element of a class of finite index of the P -hierarchy is relatively $<_\infty$ -minimal.

In [14, Theorem 3.3] Laflamme built (under MA for σ -centered partial orderings) a special ultrafilter $u_0 \in J_{\omega^{\omega+1}}^*$ the only RK-predecessor of which is a Ramsey ultrafilter. In the proof of [18, Theorem 3.13] it is shown that $u_0 \in \mathcal{P}_3$ and that U_0 is not in the form of a contour, note also that Laflamme's ultrafilter u_0 is different then ultrafilter build in Theorem 4.4 which is a contour. Therefore, we have:

Theorem 4.9. *It is consistent with ZFC that there exists a relatively-RK-3-minimal ultrafilter that is not in the form of a contour.*

5 Generic existence

In this section, the P -hierarchy is understood as $\bigcup_{1 < \alpha < \omega_1} \mathcal{P}_\alpha$.

Let V be a cascade. Denote $r_\alpha(V) = \{v \in V : r(v) = \alpha\}$. Let h be a function with the domain V such that $h(v) \in \omega$ for $v \in \max V$, and $h(v)$ is a filter on the set ω , otherwise. We define $\int^h V$ inductively as follows: $\int^h v^\uparrow$ is a principal ultrafilter generated by $h(v)$, for $v \in \max V$. If $\int^h w^\uparrow$ is defined for all $v_n \in v^\uparrow$ then $\int^h v^\uparrow = \int_{h(v)} \int v_n^\uparrow$.

Recall that the family \mathcal{F} of functions $\omega \rightarrow \omega$ is a *dominating family* if for each function $g : \omega \rightarrow \omega$ there exists $f \in \mathcal{F}$ such that $f(n) \geq g(n)$ for almost all $n < \omega$, i.e. there is n_0 such that $f(n) \geq g(n)$ for all $n > n_0$. The *dominating number* \mathfrak{d} is the minimum of cardinalities of dominating families, and \mathfrak{c} is the cardinality of the continuum.

We say that filters belonging to the family \mathbb{F} *exist generically* if each filterbase of size less than \mathfrak{c} can be extended to a filter belonging to \mathbb{F} .

In [2] Brendle showed that:

Theorem 5.1. *[2, part of Theorem E] The following are equivalent:*

- (a) $\mathfrak{d} = \mathfrak{c}$;
- (b) *ordinal ultrafilters exist generically.*

We obtain the same result for the P-hierarchy. For this we need to prove the following theorem.

Theorem 5.2. *For each ordinal $1 < \alpha < \omega_1$ and for each monotone sequential contour \mathcal{V} of rank α , the minimum of cardinalities of filterbases of \mathcal{V} is \mathfrak{d} .*

Proof. First, we will show that there exists a base of cardinality \mathfrak{d} . Let $\mathcal{V} = \int V$ for a monotone sequential cascade V . Let $\mathbb{D} = \{d_\beta : \beta < \mathfrak{d}\}$ be a dominating family for functions $V \rightarrow \omega$ (there exists such a family since V is countable). Define family $\mathbb{F} = \{f_{d,n} : d \in \mathbb{D}, n < \omega\}$ as follows: $f_{d,n}(v) = d(v)$ for $v \neq \emptyset_V$ and $f_{d,n}(v) = n$ for $v = \emptyset_V$. For each $v \in \max V$ let $(\emptyset_V = v^{0,v} \sqsubseteq v_{1,v} \sqsubseteq \dots \sqsubseteq v_{n(v),v} = v)$ be a branch of V with maximal node v . For each function $f : V \rightarrow \omega$ define sets $V(f) \subset \max V$ by the condition: $v \in V(f)$ if and only if for each i there exists k such that $v^{i+1,v} = v_k^{i,v}$ and $k \geq f(v^{i,v})$. A typical base of the cascade V is as follows: $\{V(g) : g : V \rightarrow \omega\}$. Take any $g : V \rightarrow \omega$. Since \mathbb{D} is a dominating family, there exists $d_{\beta_0} \in \mathbb{D}$ such that $g \leq^* d_{\beta_0}$. Thus, the set $A = \{v \in V : g(v) > d_{\beta_0}(v)\}$ is finite, so we can define $n_0 = \max \{n : A \cap (\emptyset_V)_n^\uparrow \neq \emptyset\} + 1$. Therefore, $V(f_{d_{\beta_0}, n_0}) \subset V(g)$, thus $\{V(f) : F \in \mathbb{F}\}$ is a base.

Now, let us assume that there exists a base $\mathbb{B} = \{B_\beta : \beta < \gamma\}$ of \mathcal{V} with $\gamma < \mathfrak{d}$. Since $\{V(f) : f : V \rightarrow \omega\}$ constitutes the base for \mathcal{V} , for each $\beta < \gamma$ there exists f_β such that $f_\beta : V \rightarrow \omega$ and $V(f_\beta) \subset B_\beta$. Let $\mathbb{G} = \{f_\beta : \beta < \gamma\}$. Since $\text{card}(\mathbb{G}) < \mathfrak{d}$, for each $v \in V$ such that $r(v) = 2$ the family $\{f_\beta \upharpoonright_{v^+} : \beta < \gamma\}$ is not a dominating family on the set v^+ . For each v such that $r(v) = 2$, take a function $g_v : v^+ \rightarrow \omega$ such that $g_v \not\leq^* f_\beta \upharpoonright_{v^+}$ for each $\beta < \gamma$. Now, let $g : V \rightarrow \omega$ be a function that $g(v) = g_{\tilde{v}}(v)$ if $r(v) = 1$ and $v \in \tilde{v}^+$; otherwise, $g(v) = 1$. We have $V(g) \in \int V = \mathcal{V}$ and $V(g) \not\subset V(f_\beta)$ for each $\beta < \gamma$. ■

The *supercontour* is a filter of type $\bigcup_{\alpha < \omega_1} \mathcal{V}_\alpha$, where $(\mathcal{V}_\alpha)_{\alpha < \omega_1}$ is an increasing sequence of monotone sequential contours such that $r(\mathcal{V}_\alpha) = \alpha$.

Corollary 5.3. *Generic existence of the P-hierarchy is equivalent to $\mathfrak{d} = \mathfrak{c}$.*

Proof. By the Ketonen's theorem [13, Theorem 1.1] generic existence of P-points is equivalent to $\mathfrak{d} = \mathfrak{c}$. By Proposition 2.2 P-points belong to \mathcal{P}_2 class; thus, if $\mathfrak{d} = \mathfrak{c}$, then the P-hierarchy generically exists.

Let $\mathfrak{d} < \mathfrak{c}$, and take an increasing ω_1 -sequence (\mathcal{V}_α) of monotone sequential contours such that $r(\mathcal{V}_\alpha) = \alpha$. Let \mathbb{B}_α be a base of cardinality \mathfrak{d} of \mathcal{V}_α (there exist by Theorem 5.2). Let $\mathbb{B} = \bigcup_{\alpha < \omega_1} \mathbb{B}_\alpha$. Obviously, $\text{card}(\mathbb{B}) = \mathfrak{d}$ and \mathbb{B} is the base for a supercontour, so it cannot be extended to any element of the P-hierarchy. ■

By Ketonen's Theorem [13] each ultrafilterbase of cardinality less than \mathfrak{d} is the base of a P-point. In order to obtain a similar result for other classes, we need the extra assumptions that such a base can be extended to infinitely many ultrafilters. To prove this we need to quote the following two results:

Theorem 5.4. [1, Theorem 4.1] *The $J_{\omega_2}^*$ -ultrafilters are the P-point ultrafilters.*

We say that families u and o *mesh* (and we write $u \# o$) whenever $U \cap O \neq \emptyset$ for every $U \in u$ and $O \in o$.

Proposition 5.5. *The following statements are equivalent:*

- a) *For each successor ordinal $1 < \alpha < \omega_1$ each filterbase of cardinality less than \mathfrak{c} which can be extended to infinitely many ultrafilters, can also be extended to some elements of \mathcal{P}_α ;*
- b) *For each successor ordinal $1 < \alpha < \omega_1$ each filterbase of cardinality less than \mathfrak{c} which can be extended to infinitely many ultrafilters, can also be extended to some elements of $J_{\omega_\alpha}^*$;*
- c) $\mathfrak{d} = \mathfrak{c}$.

Proof. For $\mathfrak{d} < \mathfrak{c}$ a proof is analogical to the second part of the proof of Theorem 5.3, with an additional use of Proposition 2.8 for the case of ordinal ultrafilters.

Now let $\mathfrak{d} = \mathfrak{c}$, and let \mathbb{B} be a proper filterbase of cardinality $< \mathfrak{c}$. By the Ketonen's Theorem [13] \mathbb{B} can be extended to a P-point, and so we can assume that $\alpha > 2$. by the assumption, there exists a family $\{F_n\}_{n < \omega}$ of pairwise disjoint sets such that $F_n \# \mathbb{B}$ for each $n < \omega$. Let (p_n) be a sequence of P-points such that $\mathbb{B} \cup \{F_n\} \subset p_n$. Take a monotone sequential cascade V of rank α . Put $R = \{v \in V : r(v) = 1\}$ and without loss of generality assume that for each $v \notin R$ a cascade v^\uparrow has no branches of length 1. Let g be an arbitrary bijections $g : R \rightarrow \omega$ and let $f_v : v^\uparrow \rightarrow F_n$ be an arbitrary bijection for each $v \in R$. Let h be a function which domain is V , defined as follows:

$$\begin{aligned} h(v') &= f_v(v') \text{ for } v' \in v^\uparrow, v \in R \\ h(v) &= p_{g(v)} \text{ for } v \in R \end{aligned}$$

$h(v) = p_1$ for other $v \in V$.

Consider $\int^h V$ and note following facts:

- 1) $\int^h V \# \mathbb{B}$, since $p_n \# \mathbb{B}$ for all n ;
- 2) $\int^h V \in \mathcal{P}_\alpha$, inductively by Theorem 2,5;
- 3) $\int^h V \in J_{\omega_{\alpha+1}}^*$ by Corollary 5.4 and inductively by Theorem 4.8. ■

6 Existence

We say that a cascade V is *built by destruction of nodes of rank 1* in a cascade W of rank $r(W) \geq 2$ iff for a set $R = \{w \in W : w(w) = 1, r(w^-) = 2\}$ there is: $V = W \setminus R$ and if $v \in R^{-W}$ then $v^{+V} = (v^{+W} \setminus R) \cup (v^{+W} \setminus R^+)$, i.e. order on the cascade is unchanged.

Observe that if W is a monotone sequential cascade then V is also a monotone sequential cascade and if $r(W)$ is finite then $r(V) = r(W) - 1$, if $r(W)$ is infinite, then $r(V) = r(W)$.

Assume that we are given a cascade of rank α and an ordinal $1 < \beta \leq \alpha$. We shall describe an operation of *decreasing the rank* of a cascade W . The construction is inductive:

For finite α , we can decrease rank of W from α to β by applying $\alpha - \beta$ times an operation of destroying nodes of rank 1 (u.e. if $\alpha = \beta$ then the cascade is unchanged).

For infinite α . Suppose that for each pair (δ, γ) where $1 < \delta \leq \gamma < \alpha$, and for each cascade W of rank γ the operation of decreasing of the rank of W from γ to δ is defined. Let W be a monotone sequential cascade of rank α , let (β_n) be a nondecreasing sequence of ordinals such that: $\beta_n = 0$ if and only if $r(W_n) = 0$, $\beta_n \leq r(W_n)$ and $\lim_{n \rightarrow \infty} (\beta_n + 1) = \delta$. Let, for each $n < \omega$, V_n be the cascade obtained by decreasing of rank of W_n to β_n . Finally let $V = (n) \leftrightarrow V_n$.

Clearly for infinite α the operation of decreasing of rank is not defined uniquely. Observe also that the above described decreasing of rank of a cascade W does not change $\max W$. If a cascade V is obtained from W by decreasing of rank, then we write $V \triangleleft W$. Trivially $V \triangleleft W$ and inductively $\int V \subset \int W$.

Theorem 6.1. [5] *If $(\mathcal{V}_n)_{n < \omega}$ is a sequence of monotone sequential contours of rank less than α and $\bigcup_{n < \omega} \mathcal{V}_n$ has the finite intersection property, then there is no monotone sequential contour \mathcal{W} of rank $\alpha + 1$ such that $\mathcal{W} \subset \langle \bigcup_{n < \omega} \mathcal{V}_n \rangle$.*

Before we prove the main Lemma we shall prove a technical claim;

Lemma 6.2. *Let V be the cascade of rank α , W be cascade obtained from V by decreasing the rank of V to $\beta < \alpha$ and let $\beta < \gamma < \alpha$. Then there is a cascade T of rank γ such that $W \triangleleft T \triangleleft V$.*

Proof. If $\beta = 1$ then it suffice to take any monotone sequential cascade T obtained by decreasing of the rank of W to γ .

If $\beta > 1$ then take (β_n) - a nondecreasing sequence of ordinals such that: $\beta_n = 0$ if and only if $r(W_n) = 0$, $r(V_n) \leq \beta_n \leq r(W_n)$ and $\lim_{n \rightarrow \infty} (\beta_n + 1) = \gamma$. By inductive assumption one can find (T_n) a sequence of monotone sequential contours such that $V_n \triangleleft T_n \triangleleft W_n$. Put $T = (n) \leftarrow^{\rho} T_n$. ■

We write $V \triangleleft_1 W$ if $\max W \in \int V$ and $V \downarrow^{\max W} \triangleleft W$. We write $V \triangleleft_2 W$ if $\max V \in \int W$ and $V \triangleleft W \downarrow^{\max V}$. Trivially Lemma 6.2 is true also for \triangleleft_1 , \triangleleft_2 instead of \triangleleft .

Lemma 6.3. *Let $\alpha < \omega_1$ be a limit ordinal and let $(\mathcal{V}^n : n < \omega)$ be a sequence of monotone sequential contours such that $r(\mathcal{V}^n) < r(\mathcal{V}^{n+1}) < \alpha$ for every n and such that $\bigcup_{n < \omega} \mathcal{V}^n$ has the finite intersection property. Then there is no monotone sequential contour \mathcal{W} of rank α such that $\mathcal{W} \subset \langle \bigcup_{n < \omega} \mathcal{V}^n \rangle$.*

Proof. Put $\alpha_n = r(\mathcal{V}^n)$, without loss of generality we may assume that $\alpha_1 \geq 3$. Assume that there exists a monotone sequential contour \mathcal{W} of rank α such that $\mathcal{W} \subset \langle \bigcup_{n < \omega} \mathcal{V}^n \rangle$. We build a cascade W and a sequence of cascades $(W^n)_{n < \omega}$ such that:

- $\int W = \mathcal{W}$;
- $W^n \triangleleft_1 W^{n+1}$ for all n ;
- $W^n \triangleleft_2 W$ for all n ;
- $r(W^n) = \alpha_n + 3$ for all n ;
- $r(W_i^n) = \alpha_n + 2$ for all n and all i ;
- $r(W_{i,j}^n) = \alpha_n + 1$ for all n, i and j .

Fix any monotone sequential cascade \bar{W} such that $\int \bar{W} = \mathcal{W}$. Let \bar{W}^m be the cascade obtained from \bar{W} by cutting every subcascade \bar{W}_i of rank smaller than $\alpha_m + 2$ and every subcascade $\bar{W}_{i,j}$ of rank smaller than $\alpha_m + 1$. Observe that we cut only finitely many subcascades \bar{W}_i and for the other \bar{W}_i only finitely many subcascades $\bar{W}_{i,j}$. Thus $\int \bar{W}^m = \int \bar{W} = \mathcal{W}$ for every m .

Let $W = \bar{W}^1$ and W_1 be a cascade obtained from \bar{W}_1 by decreasing ranks of $W_{i,j}^1$ to $\alpha_1 + 1$. Thus $W^1 \triangleleft_2 W$.

Since cascades \bar{W}^n and W^n are subcascades of W thus for nodes (and so subcascades) of \bar{W}^n and W^n we may keep the indexation from W , to avoid the collision of notation we put those indexes in parenthesis.

Assume that $W^1 \blacktriangleleft W^2 \blacktriangleleft \dots \blacktriangleleft W^m$ have been defined. We apply Lemma 6.2 to cascades $W_{(i,j)}^m$ and $\bar{W}_{(i,j)}^{m+1}$ to define $W_{(i,j)}^{m+1}$ of rank $\alpha_{m+1} + 1$ for those (i, j) that $w_{i,j} \in W^m \cap \bar{W}^{m+1}$.

Let K^{m+1} be a subcascade of W with $K^{m+1} = \{\emptyset_W\} \cup (\emptyset_W)^{+\bar{W}^{m+1}} \cup ((\emptyset_W)^{+\bar{W}^{m+1}})^{+\bar{W}^{m+1}}$. Put $W^{m+1} = K^{m+1} \leftarrow W_{(i,j)}^{m+1}$.

Next we build a decreasing sequence $(U_n)_{n < \omega}$ satisfying conditions U_A - U_D :

1. $U_A(n)$: $U_n \in \int W^n$;
2. $U_B(n)$: $U_n \notin \langle \bigcup_{i \leq n} \mathcal{V}^i \rangle$;
3. $U_C(n)$: $U_n \cap (\omega \setminus \max W^{n+1}) = U_{n+1} \cap (\omega \setminus \max \bar{W}^{n+1})$;
4. $U_D(n)$: $U_n \cap \max W_i \in \int W_i$ for all n and all i .

In this aim first we built an additional sequence (\widetilde{W}^n) of cascades by $\widetilde{W}^n = W^n \setminus \emptyset_{W^n}^+$ such that $\emptyset_{W^n}^+ = \bigcup \{w^+ : w \in \emptyset_{W^n}^+\}$. and that the rest of cascades we leave unchanged (we may say that \widetilde{W}^n is obtained from W^n by destroying all nodes of rank $\alpha_m + 2$). Notice that \widetilde{W}^n is a monotone sequential cascade of rank $\alpha_m + 2$, and that if a set U_n fulfills conditions $U_B(n)$, $U_C(n)$ and belongs to \widetilde{W}^n then the same set U_n fulfills all conditions $U_A(n) - U_D(n)$.

Put $U_0 = \omega$. Assume that U_0, U_1, \dots, U_{n-1} was defined, but it is impossible to define U_n . This means that every set $U \in \int \widetilde{W}_n$ is contained in $\langle \bigcup_{i < n} \mathcal{V}_i \rangle$. On the other side $\max \widetilde{W}_n \in \mathcal{W}$ and so the family $\{U \cap \max \widetilde{W}_n : U \in \bigcup_{i \leq n} \mathcal{V}_i\}$ has the finite intersection property. By the theorem of Dolecki $\langle \{U \cap \max \widetilde{W}_n : U \in \bigcup_{i \leq n} \mathcal{V}_i\} \rangle$ do not contain any monotone sequential contour of rank $\alpha_n + 2$ and so do not contain $\int \widetilde{W}_n$. A contradiction. On each step of induction we can put $\bigcap_{i \leq n} U_i$ instead of U_n and assume that the sequence $(U_n)_{n < \omega}$ is decreasing.

Notice that $\bigcup_{n < \omega} (\max W_{n+1})^c = \max W$, let $U = \bigcap_{n < \omega} U_n$. Conditions (1)-(4) guarantee that

- 1) $U \in \int W$ and
- 2) $U \notin \langle \bigcup_{n < \omega} \mathcal{V}_n \rangle$.

To see 1) fix any $t < \omega$, note that $\max W^m \in \int W_t$ only for finite number of m . So the sequence $(U_n \cup R)_{n < \omega}$ is (decreasing and) almost constant on some $R \in \int W_t$. Therefore $\bigcap_{n < \omega} U_n \cap R$ is indeed a finite intersection of R and U_n all of which by condition U_D belongs to $\int W_t$. So $\bigcap_{n < \omega} U_n \in \int W_t$ for all t , and so $U \in \int W$.

To see 2), assume that $U \in \langle \bigcup_{n < \omega} \mathcal{V}_n \rangle$, then there is a finite $M < \omega$ such that $U \in \langle \bigcup_{n < M} \mathcal{V}_n \rangle$. But $U_M \notin \langle \bigcup_{n \leq M} \mathcal{V}_n \rangle$ and $U \subset U_M$. Thus $U \notin \langle \bigcup_{n < \omega} \mathcal{V}_n \rangle$. A contradiction. ■

Proposition 6.4. [18, part of corollary 2.6] (ZFC) Classes \mathcal{P}_1 and \mathcal{P}_{ω_1} are nonempty.

Theorem 6.5. (CH) Each class of the P -hierarchy is nonempty.

Proof. For successor α 's and for 1 for ω_1 we deal in Proposition 6.4. By well known result of W. Rudin CH implies existing of P -points so for successor α we are done by Theorem 6.5. Let $\alpha < \omega_1$ be limit ordinal. Let (\mathcal{V}_n) be an increasing sequence of monotone sequential contours such that $r(\mathcal{V}_n)$ is an increasing sequence with $\lim_{n < \omega} r(\mathcal{V}_n) = \alpha$. By CH we can order all α -partitions in an ω_1 sequence (P_β) .

We will build a sequence $(Q_\beta)_{\beta < \omega_1}$ of subsets of ω such that Q_β is residual for the partition P_β and a family $\{Q_\beta : \beta < \omega_1\} \cup \bigcup_{n < \omega} \mathcal{V}_n$ has the finite intersection property. Since $\bigcup_{n < \omega} \mathcal{V}_n$ is a filter and, by the Lemma 6.3 above, does not contain any monotone sequential contour of rank α , thus there exists a set Q_1 residual for the partition P_1 such that the family $\{Q_1\} \cup \bigcup_{n < \omega} \mathcal{V}_n$ has the finite intersection property. Suppose now that the sequence $(Q_\beta)_{\beta < \gamma}$ is already built. If $\gamma < \omega$ then consider the sequence $(\mathcal{V}_n \upharpoonright_{\bigcap_{\beta < \gamma} Q_\beta})_{n < \omega}$, this is an increasing sequence of monotone sequential contours with $r(\mathcal{V}_n) = r(\mathcal{V}_n \upharpoonright_{\bigcap_{\beta < \gamma} Q_\beta})$ thus by the Lemma 6.3 there exist a set Q_γ residual for the partition P_γ and such that a family $\{Q_\gamma\} \cup \bigcup_{n < \omega} (\mathcal{V}_n \upharpoonright_{\bigcap_{\beta < \gamma} Q_\beta})$ has the finite intersection property and thus also a family $\{Q_\beta : \beta \leq \gamma\} \cup \bigcup_{n < \omega} \mathcal{V}_n$ has the finite intersection property. If $\gamma \geq \omega$ then we enumerate the sequence $(Q_\beta)_{\beta < \gamma}$ by natural numbers and obtain the sequence $(Q^{\gamma, n})_{n < \omega}$. Consider the sequence $(\mathcal{V}_n \upharpoonright_{\bigcap_{m \leq n} Q^{\gamma, m}})_{n < \omega}$, this is an increasing sequence of monotone sequential contours with $r(\mathcal{V}_n) = r(\mathcal{V}_n \upharpoonright_{\bigcap_{m \leq n} Q^{\gamma, m}})$. Thus by the Lemma 6.3 there exist a set Q_γ residual for the partition P_γ and such that a family $\{Q_\gamma\} \cup \bigcup_{n < \omega} (\mathcal{V}_n \upharpoonright_{\bigcap_{m \leq n} Q^{\gamma, m}})$ has the finite intersection property and thus also a family $\{Q_\beta : \beta \leq \gamma\} \cup \bigcup_{n < \omega} \mathcal{V}_n$ has the finite intersection property. Thus a sequence $(Q_\beta)_{\beta < \omega_1}$ with described properties exists.

Now it is sufficient to take any ultrafilter u that contains $\{Q_\beta : \beta < \omega_1\} \cup \bigcup_{n < \omega} \mathcal{V}_n$. Since $\bigcup_{n < \omega} \mathcal{V}_n \subset u$ then u contains a monotone sequential contour of each rank less than α . Since u contains $\{Q_\beta : \beta < \omega_1\}$ thus u contains

residual set for each α -partition, and thus u do not contain any monotone sequential contour of rank α . ■

Notice that it was also shown

Theorem 6.6. *[18, reformulation of Theorem 3.12]*

$MA_{\sigma\text{-center.}}$ implies $\mathcal{P}_{\alpha+\omega} \neq \emptyset$.

It is worth to compare the above results with [1, Theorem 4.2], where Baumgartner proved that if P-points exist then for each successor $\alpha < \omega_1$ the class of $J_{\omega^\alpha}^*$ ultrafilters is nonempty, and with our theorem from [17] where we proved (in ZFC) that a class of $J_{\omega^\omega}^*$ ultrafilters is empty.

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